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Contextual extensions of C^* algebras and hidden variable theories

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Abstract. A general algebraic formalism for studying contextual hidden variable theories is developed. The contextuality is understood as a manifestation of an inadequacy in the algebra of quantum observables for the complete description of the system. It is shown that it is always possible to 'improve' the algebra of quantum observables, explicitly paying attention to contexts, such that this improved algebra becomes a base for a hidden variable theory.

1. Introduction

The main motivation for this work comes from hidden variable (HV) theories which are, at the statistical level, in agreement with quantum mechanics [1, 5, 8].

All these theories are causal. Their common philosophy is that the quantum description of the world is not complete. The appearance of probabilities in quantum mechanics is interpreted as a consequence of this incompleteness.

Each hidden variable theory deals with *subquantum states* that are, by definition, complete states of the system (in this theory). If a subquantum state is fixed, the outcome of any quantum measurement is determined. Consequently, it is possible to speak about values of quantum observables in subquantum states. On the other hand, the quantum states appear as certain *probability measures* on the space of all subquantum states (subquantum space). Accordingly, there exists an analogy between the relation: HV theory⇔quantum mechanics and the relation: classical mechanics⇔ classical statistical mechanics. The subquantum space corresponds to the phase space in classical mechanics.

In contrast to this analogy, HV theories necessarily have *contextual* features: the value of a given observable in a given subquantum state is defined only after specification of the so-called *measurement context* [13]. An important fact about contexts is that they are in some sense external, relative to the algebra of quantum observables. The same observable could have different values, in a given subquantum state, in different contexts.

We can understand the contextuality as a necessary consistency condition because it can be shown [3, 7, 11] that a HV theory in which the mentioned values are determined only by subquantum states does not exist.

Therefore, before developing some hidden variable theory, one must specify entities representing contexts. In the case of the standard quantum mechanical structure $(H, P(H))$ (that is, pure states are represented by vectors in a separable Hilbert space H, quantum events by elements of the projector lattice $P(H)$) it is supposed that each

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quantum measurement is a *minimal* measurement [14] of some observable. In this situation, it is natural to identify the corresponding context with the range of the spectral measure of this observable.

This picture can be taken as starting point for various generalizations. The quantum logical one can be obtained by recognizing contexts as Boolean sub-σ-algebras of $P(H)$ and replacing the projector lattice $P(H)$ by some weaker structure (see [8]). We can also proceed algebraically [5], recognizing contexts in the case of the standard quantum structure as commutative von Neumann subalgebras of the algebra $L(H)$ of all bounded operators, because in the case of a separable Hilbert space H there is a natural bijection (bicomutant):

$$\{\text{Boolean sub-}\sigma\text{-algebras of } P(H)\}$$
$$\to \{\text{commutative von Neumann subalgebras of } L(H)\}.$$

In this paper, we develop a general C^* algebraic formalism for treating contextual hidden variables. We start from some C^* algebra Σ (algebra of 'quantum observables') and some family $\boldsymbol{T}$ of commutative C^*-subalgebras of Σ ('measurement contexts'). The possibility that observable $\hat{a} \in \Sigma$ shows, on the subquantum level, different faces if we use different contexts $A_1, A_2 \in \boldsymbol{T}$, $\hat{a} \in A_1, A_2$, is understood here as an *inadequacy* of the C^* algebra Σ for the complete description of the system. This thinking naturally leads to the idea of replacing the algebra Σ by some 'finer' algebra Σ' in such a way that contexts are explicitly taken into account.

In section 2 we formalize this idea of contextual refinements introducing a notion of contextual extension. We consider the collection of all contextual extensions of a given pair $(\Sigma, \boldsymbol{T})$ and show that there exists 'the biggest' one. The corresponding C^* algebra (denoted by $Ctx(\Sigma, \boldsymbol{T})$) will play an important role throughout the paper.

In section 3 we discuss problems related to hidden variables. From the point of view of hidden variables, not all contextual extensions are relevant. Roughly speaking, we have to restrict ourselves to contextual extensions in which all 'quantum states' are reducible to some mixtures of 'subquantum states'. The precise formulation of this property leads to a notion of HV extension (definition 3.1). We show that algebra $Ctx(\Sigma, \boldsymbol{T})$ naturally gives rise to a HV extension of $(\Sigma, \boldsymbol{T})$. We then investigate some properties of the collection of all HV extensions of $(\Sigma, \boldsymbol{T})$.

We shall deal only with C^* algebras with unity and with unity-preserving homomorphisms.

2. Contextual extensions

To begin with, let us fix a C^* algebra Σ and a family $\boldsymbol{T}$ of commutative C^* subalgebras of Σ. We shall suppose only that $\boldsymbol{T}$ generate Σ. For $A \in \boldsymbol{T}$ we denote by $i_A : A \to \Sigma$ canonical inclusion.

Definition 2.1. A contextual extension of $(\Sigma, \boldsymbol{T})$ is a triplet $(\Sigma', \phi, \{\iota_A : A \in \boldsymbol{T}\})$ where

(i) Σ' is a C^* algebra with unity;

(ii) $\phi : \Sigma' \to \Sigma$ is a C^* homomorphism;

(iii) $\{\iota_A : A \in \boldsymbol{T}\}$ is a family C^* of homomorphisms $\iota_A : A \to \Sigma'$ such that $\phi\iota_A = i_A$ for any $A \in \boldsymbol{T}$;

(iv) a family $\{\iota_A(A) : A \in \boldsymbol{T}\}$ generate Σ'.

As an immediate consequence of this definition, and of the fact that the image of a C^* homomorphism is closed [4], we obtain the following.

Proposition 2.1. Let $(\Sigma', \phi, \{\iota_A: A \in \boldsymbol{T}\})$ be a contextual extension of $(\Sigma, \boldsymbol{T})$. Then ϕ is surjective and maps isomorphically $\iota_A(A)$ onto A, for any $A \in \boldsymbol{T}$.

Now, we are going to introduce an important example of a contextual extension. Let $Ctx(\Sigma, \boldsymbol{T})$ be the C^* algebra generated by the set of elements $S = \{(\hat{a}, A): A \in \boldsymbol{T}, \hat{a} \in A\}$ and the following relations:

$$\begin{aligned}
&(1, A) = 1 \\
&\alpha(\hat{a}, A) + \beta(\hat{b}, A) = (\alpha\hat{a} + \beta\hat{b}, A) \qquad \alpha, \beta \in C \\
&(\hat{a}, A)(\hat{b}, A) = (\hat{a}\hat{b}, A) \\
&(\hat{a}, A)^* = (\hat{a}^*, A).
\end{aligned} \tag{1}$$

More precisely, we consider first the free * algebra generated by elements of S. Then we factorize through the two-sided * ideal generated by relations (1). Let K be a * algebra obtained in this way. A seminorm $p: K \to R^+$ is called admissible iff there is a * representation $\pi: K \to L(H)$ of K by bounded operators in some Hilbert space H such that $p(x) = \|\pi(x)\|_H$ for all $x \in K$. We define a seminorm $\| \ \|$ on K: $\|x\| = \sup_p\{p(x)\}$, where the supremum is taken over the set of all admissible seminorms. (It can easily be shown that $\|x\|$ is finite for each $x \in K$). Obviously, the characteristic C^* property of this seminorm $\|x^*x\| = \|x\|^2$ is fulfilled. To get a norm, we consider two-sided ideal $N = \{x \in K: \|x\| = 0\}$. On the factor algebra K/N there is a natural norm, induced by $\| \ \|$. Finally, we complete K/N and get C^* algebra $Ctx(\Sigma, \boldsymbol{T})$.

By construction, any $A \in \boldsymbol{T}$ gives rise to a C^* homomorphism $\hat{\iota}_A: A \to Ctx(\Sigma, \boldsymbol{T})$: $\hat{\iota}_A(a) = [(a, A)]$. Everywhere dense * subalgebra K/N of $Ctx(\Sigma, \boldsymbol{T})$ coincides with the * subalgebra generated by elements of the form $\hat{\iota}_A(a)$.

Proposition 2.2. (i) Let Σ' be a C^* algebra and suppose that for each $A \in \boldsymbol{T}$ a * homomorphism $\lambda_A: A \to \Sigma'$ is defined. Then there exists one, and only one, * homomorphism $\lambda: Ctx(\Sigma, \boldsymbol{T}) \to \Sigma'$ such that $\lambda_A = \lambda\hat{\iota}_A$ for every $A \in \boldsymbol{T}$.

(ii) If $\{\lambda_A(A): A \in \boldsymbol{T}\}$ generate algebra Σ', then λ *is surjective.*

Proof. The existence and uniqueness of λ is an immediate consequence of the definition of $Ctx(\Sigma, \boldsymbol{T})$ and the fact that each C^* algebra can be realized as a C^* algebra of bounded operators in some Hilbert space. (For example, take the direct sum of all cyclic representations). If $\{\lambda_A(A): A \in \boldsymbol{T}\}$ generate Σ', then the image of λ is everywhere dense in Σ'. On the other hand, it is closed as an image of a C* homomorphism. Thus, λ is surjective. □

Example 1. Let $\Sigma' = \Sigma$ and $\lambda_A = i_A$. Proposition 2.3 implies that there exists one, and only one, surjective * homomorphism $\hat{\phi}: Ctx(\Sigma, \boldsymbol{T}) \to \Sigma$ such that $\hat{\phi}(\hat{\iota}_A(\hat{a})) = \hat{a}$.

Example 2. For each $A \in \boldsymbol{T}$ consider the spectrum spec(A). Algebra A is then naturally isomorphic to the algebra of all complex valued continuous functions on spec(A). Let $\Pi_{\boldsymbol{T}} = \Pi_{A \in \boldsymbol{T}}$ spec(A), and endow it with the Tihonov product topology. In this topology, $\Pi_{\boldsymbol{T}}$ is a compact topological space. Condition (i) of proposition 2.2 is satisfied by putting $\Sigma' = C(\Pi_T)$ and $\lambda_A(\hat{a})(x) = \hat{a}(\pi_A(x))$. Here, $\pi_A : \Pi_T \to \mathrm{spec}(A)$ is the Ath coordinate projection and $\hat{a}$ is considered here as a function on spec(A). Condition (ii) is also satisfied because * algebra generated by functions $\lambda_A(\hat{a})$ distinguishes points of Π_T and, according to the Weierstrass–Stone theorem, it is everywhere dense in $C(\Pi_T)$. Therefore, there exists a natural surjective C^* homomorphism $\hat{s} : Ctx(\Sigma, \boldsymbol{T}) \to C(\Pi_T)$.

As a simple consequence of proposition 2.2 and example 1 we obtain the following.

Proposition 2.3. (i) The triplet $(Ctx(\Sigma, \boldsymbol{T}), \hat{\phi}, \{\hat{\iota}_A : A \in \boldsymbol{T}\})$ is a contextual extension of $(\Sigma, \boldsymbol{T})$.

(ii) If $(\Sigma', \phi, \{\iota_A : A \in \boldsymbol{T}\})$ be a contextual extension of $(\Sigma, \boldsymbol{T})$ then there exists one, and only one, * homomorphism $\lambda' : Ctx(\Sigma, \boldsymbol{T}) \to \Sigma'$ such that $\iota_A = \lambda' \hat{\iota}_A$, for every $A \in \boldsymbol{T}$. The map λ' is surjective.

It is of some interest to introduce a category $CTX(\Sigma, T)$ of all contextual extensions of $(\Sigma, \boldsymbol{T})$. Objects in this category are contextual extensions and we define a morphism of two objects $(\Sigma_1, \phi_1, \{\iota^1_A : A \in \boldsymbol{T}\})$ and $(\Sigma_2, \phi_2, \{\iota^2_A : A \in \boldsymbol{T}\})$ as a * homomorphism $\lambda_{1,2} : \Sigma_1 \to \Sigma_2$ with the property $\iota^2_A = \lambda_{1,2}\iota^1_A$, for every $A \in \boldsymbol{T}$. Such a homomorphism is necessarily surjective. If it exists, it is unique.

The category $CTX(\Sigma, \boldsymbol{T})$ possesses the first and the last object: The first object is, according to proposition 2.3, the triplet $(Ctx(\Sigma, \boldsymbol{T}), \hat{\phi}, \{\hat{\iota}_A : A \in \boldsymbol{T}\})$. The last object is the triplet $(\Sigma, id, \{i_A : A \in \boldsymbol{T}\})$.

3. Contextual hidden variables

In this section we analyse hidden variables from the point of view of contextual extensions. A given contextual extension $(\Sigma', \phi, \{\iota_A : A \in \boldsymbol{T}\})$ becomes a base for some HV theory only if each state on Σ can be reduced to a mixture of 'subquantum states'. These states, of course, should enter the game via the algebra of 'right' observables Σ'.

'Subquantum states', as complete states of the system, can be characterized by the property of having zeroth dispersion in any observable from Σ'. It is easy to see that *characters* (non-trivial multiplicative *-functionals) are the only states of this kind.

Let $\Omega(\Sigma')$ denote the set of all characters of Σ', endowed with the *-weak topology. The space $\Omega(\Sigma')$ is compact. Let $k(\Sigma')$ be the ideal generated by commutators. There is a natural surjective C^* homomorphism $e' : \Sigma' \to C[\Omega(\Sigma')]$ defined by $e'(a)(x) = x(a)$.

Example 3. It is easy to see that $\Omega[Ctx(\Sigma, \boldsymbol{T})] = \Pi_T$.

Definition 3.1. A contextual extension $(\Sigma', \phi, \{\iota_A : A \in \boldsymbol{T}\})$ of $(\Sigma, \boldsymbol{T})$ is called an HV *extension* iff for any state ρ on Σ there exist at least one state ρ' on Σ' such that:

(i) $\rho'|_{k(\Sigma')} = 0$;

(ii) For any $A \in \boldsymbol{T}$ and $\hat{a} \in A$, $\rho'(\iota_A(\hat{a})) = \rho(\hat{a})$.

Remark (justification of definition 3.1). Let $(\Sigma', \phi, \{\iota_A : A \in \boldsymbol{T}\})$ be an HV extension. As we already mentioned, elements of the space $\Omega(\Sigma')$ are interpreted as possible complete states of the system. It is natural to introduce, for a given $\omega \in \Omega(\Sigma')$, $A \in \boldsymbol{T}$ and $\hat{a} \in A$, a *value* $F_A(\hat{a})(\omega)$ of observable $\hat{a}$ in state ω relative to context A: $F_A(\hat{a})(\omega) = \omega(\iota_A(\hat{a}))$.

According to definition 3.1, for any state ρ on Σ there is a state ρ' on Σ' which can be projected to $\Sigma'/k(\Sigma') = C[\Omega(\Sigma')]$ such that $\rho'(\iota_A(\hat{a})) = \rho(\hat{a})$. This equation can be rewritten in the form $\rho(\hat{a}) = \int F_A(\hat{a})(\omega)\, d\mu'(\omega)$, where the integral is taken over $\Omega(\Sigma')$ and μ' is the probability measure on the Borel σ-field of $\Omega(\Sigma')$ which, according to the Riesz theorem, corresponds to the projected ρ'. Thus, any 'quantum state' can be interpreted as some 'lack of knowledge' about variables $\omega \in \Omega(\Sigma')$.

Proposition 3.1. The triplet $(Ctx(\Sigma, \boldsymbol{T}), \hat{\phi}, \{\hat{\iota}_A: A \in \boldsymbol{T}\})$ is an HV extension.

Proof. Let ρ be a state on Σ. We know that the algebra $Ctx(\Sigma, \boldsymbol{T})$ is linearly generated by elements of the form $(\hat{a}_1, A_1)\ldots(\hat{a}_n, A_n)$. It is easy to see that the formula $\rho'[(\hat{a}_1, A_1)\ldots(\hat{a}_n, A_n)] = \rho(\hat{a}_1)\ldots\rho(\hat{a}_n)$ consistently and uniquely defines a state ρ' on $Ctx(\Sigma, \boldsymbol{T})$ with desired properties. □

The next proposition connects HV extensions with certain ideals in algebra $Ctx(\Sigma, \boldsymbol{T})$.

Definition 3.2. An ideal J in $Ctx(\Sigma, \boldsymbol{T})$ is called HV *admissible* iff:

(i) $J \subseteq \ker \hat{\phi}$;

(ii) For any state ρ on Σ there is a state $\bar{\rho}$ on $Ctx(\Sigma, \boldsymbol{T})$ such that $\bar{\rho}|_J = 0$, $\bar{\rho}|_{k[Ctx(\Sigma,\boldsymbol{T})]} = 0$ and $\bar{\rho}[(\hat{a}, A)] = \rho(\hat{a})$.

Proposition 3.2. Let $(\Sigma', \phi, \{\iota_A: A \in \boldsymbol{T}\})$ be an HV extension and $\lambda': Ctx(\Sigma, \boldsymbol{T}) \to \Sigma'$ a homomorphism introduced in proposition 2.3. Then $J = \ker \lambda'$ is an HV admissible ideal.

Conversely, let J be an HV admissible ideal. We define $\Sigma' = Ctx(\Sigma, \boldsymbol{T})/J$, a map $\lambda': Ctx(\Sigma, \boldsymbol{T}) \to \Sigma'$ as projection, a map $\phi: \Sigma' \to \Sigma$ by equality $\hat{\phi} = \phi\lambda'$ and maps $\{\iota_A: A \in \boldsymbol{T}\}$ by equality $\iota_A = \lambda'\hat{\iota}_A$. Then the triplet $(\Sigma', \phi, \{\iota_A: A \in \boldsymbol{T}\})$ is an HV extension.

Proof. Let $(\Sigma', \phi, \{\iota_A: A \in \boldsymbol{T}\})$ be an HV extension. For any state ρ on Σ there is a state ρ' on Σ' such that $\rho'|_{k(\Sigma')} = 0$, $\rho'(\iota_A(a)) = \rho(a)$. Define a state $\bar{\rho}$ on $Ctx(\Sigma, \boldsymbol{T})$ by $\bar{\rho} = \rho'\lambda'$. The map λ' is surjective. Therefore we have $\lambda' k[Ctx(\Sigma, \boldsymbol{T})] = k(\Sigma')$. We conclude that $\bar{\rho}|_J = 0$, $\bar{\rho}|_{k[Ctx(\Sigma,\boldsymbol{T})]} = 0$ and $\bar{\rho}[(\hat{a}, A)] = \rho(\hat{a})$. Thus, J is HV admissible.

Let J be an HV admissible ideal in $Ctx(\Sigma, \boldsymbol{T})$. Then any state $\bar{\rho}$ from definition 3.2 can be projected to a state ρ' on Σ' satisfying $\rho'|_{k(\Sigma')} = 0$, $\rho'(\iota_A(\hat{a})) = \bar{\rho}[(\hat{a}, A)] = \rho(\hat{a})$. We conclude that $(\Sigma', \phi, \{\iota_A: A \in \boldsymbol{T}\})$ is an HV extension. □

Proposition 3.3. Each HV admissible ideal in $Ctx(\Sigma, \boldsymbol{T})$ is contained in some maximal HV admissible ideal.

Proof. Consider an arbitrary ordered family $\{J_\alpha;\ \alpha \in I,\ \alpha < \beta \Rightarrow J_\alpha \subseteq J_\beta\}$ of HV admissible ideals in $Ctx(\Sigma, \boldsymbol{T})$. Let $J = \bigcup_{\alpha \in I} J_\alpha$. We shall prove that J is also an HV admissible ideal.

It is clear that $J \subseteq \ker \hat{\phi}$. For any state ρ on Σ and $\alpha \in I$, there is a state ρ'_α on Σ' with property $\rho'_\alpha[(\hat{a}, A)] = \rho(\hat{a})$, $\rho'_\alpha|_J = 0$, $\rho'_\alpha|_{k[Ctx(\Sigma,\boldsymbol{T})]} = 0$. The set of all states in Σ' is compact in the *-weak topology. Consequently, a hypersequence $\{\rho_\alpha;\ \alpha \in I\}$ has a convergent subsequence, which converges to some state ρ'. It is easy to see that $\rho'|_J = 0$, $\rho'|_{k[(Ctx(\Sigma,\boldsymbol{T})]} = 0$ and $\rho'[(\hat{a}, A)] = \rho(\hat{a})$. The statement of the proposition follows now from the lemma of Zorn. □

In the rest of this section we discuss properties of a collection of all HV extensions. They form a complete subcategory of $CTX(\Sigma, \boldsymbol{T})$, which we denote by $\text{HV}(\Sigma, \boldsymbol{T})$. According to proposition 3.1, this category has the initial object $(Ctx(\Sigma, \boldsymbol{T}), \hat{\phi}, \{\hat{\iota}_A: A \in \boldsymbol{T}\})$.

An object in $\text{HV}(\Sigma, \boldsymbol{T})$ is called maximal iff the only morphisms starting from it are isomorphisms. Maximal objects in $\text{HV}(\Sigma, \boldsymbol{T})$ corresponds to maximal HV admissible ideals in $Ctx(\Sigma, \boldsymbol{T})$. According to proposition 3.3, for any HV extension $(\Sigma_1, \phi_1, \{\iota_A^1: A \in \boldsymbol{T}\})$ there exists a maximal HV extension $(\Sigma_2, \phi_2, \{\iota_A^2: A \in \boldsymbol{T}\})$ and a * homomorphism $\lambda_{1,2}: \Sigma_1 \to \Sigma_2$ with the property $\iota_A^2 = \lambda_{1,2}\iota_A^1$.

The structure of maximal HV extensions is particularly simple in the case when the algebra Σ has no characters. For example, the algebra $L(H)$, where H is a Hilbert space, has this property.

Proposition 3.4. Let $(\Sigma', \phi, \{\iota_A: A \in \boldsymbol{T}\})$ be a maximal HV extension and Σ a C^* algebra without characters. Then the direct sum of maps $e' + \phi: \Sigma' \to C[\Omega(\Sigma')] + \Sigma$ is an isomorphism.

Proof. Firstly, $e' + \phi$ is always surjective. This follows from the surjectivity of e' and the fact that $(e' + \phi)[k(\Sigma')] = (0, k(\Sigma)) = (0, \Sigma)$. To prove injectivity, let us consider a map $\lambda': Ctx(\Sigma, \boldsymbol{T}) \to \Sigma'$ introduced in proposition 2.3. An ideal $J = \lambda'^{-1}[\ker(e' + \phi)] = \lambda'^{-1}[\ker e'] \cap \lambda'^{-1}[\ker \phi]$ is HV admissible and $\ker \lambda' \subseteq J$. On the other hand, $(\Sigma', \phi, \{\iota_A: A \in \boldsymbol{T}\})$ is maximal. This implies $J = \ker \lambda'$ and we conclude that $e' + \phi$ is injective. □

Finally, it is interesting to see how relations between different HV extensions are reflected at the level of 'subquantum spaces': Suppose that HV extensions $(\Sigma_1, \phi_1, \{\iota_A^1: A \in \boldsymbol{T}\})$ and $(\Sigma_2, \phi_2, \{\iota_A^2: A \in \boldsymbol{T}\})$ and a homomorphism $\lambda_{1,2}: \Sigma_1 \to \Sigma_2$ with the property $\iota_A^2 = \lambda_{1,2}\iota_A^1$ are given (a morphism in $\text{HV}(\Sigma, \boldsymbol{T})$. Then we can define a map $\lambda_{1,2}^T: \Omega(\Sigma_2) \to \Omega(\Sigma_1)$ by $\lambda_{1,2}^T(\omega)(x) = \omega[\lambda_{1,2}(x)]$, $\omega \in \Omega(\Sigma_2)$, $x \in \Sigma_1$. This map is continuous and injective, because $\lambda_{1,2}$ is surjective. In particular, any subquantum space can be naturally seen as a closed subspace of Π_T. *Maximality* of an HV extension $(\Sigma', \phi, \{\iota_A: A \in \boldsymbol{T}\})$ implies therefore *minimality* of the space $\Omega(\Sigma')$ in Π_T.

4. Conclusion

In this paper we have shown, in the full generality, that it is always possible to construct an extension Σ' of an algebra of observables Σ such that a theory formulated on Σ' is causal and interprets all stochasticity inherent in Σ via the lack-of-knowledge mechanism.

Unfortunately, if we confine the formalism presented here to the quantum mechanics and take into account an assumption of (subquantum) locality, we find it satisfactory only at the *one-particle level*: At the many-particles level we meet the problem of (non-)locality which, due to Bell's inequalities [2] can not be overcome. In other words, a local HV theory of the type of some HV extension (in the sense of definition 3.1) is not possible.

However, a more careful analysis shows that in all derivations of Bell's inequalities, besides the locality assumption, some assumption about *statistics* appears. More precisely, it is true that any local HV theory based on classical statistics satisfies Bell's inequalities. On the other side, it is not true that any local HV theory satisfies Bell's inequalities. Pitowsky [12] and Gudder [9, 10] have constructed examples of local HV models, based on a generalized statistics, which are in agreement with quantum theory.

A generalized concept of probability can be incorporated in the formalism of HV extensions. It turns out [6] that after a suitable modification, this formalism becomes explicitly local and covariant.

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